Regular Reuleaux Spheroforms

Nick Misenheimer and Nick Dudley
Department of Mathematics and Statistics
University of Arkansas at Little Rock
2801 S. University
Little Rock, AR 72204
nrmisenheime@ualr.edu, nrdudley@ualr.edu

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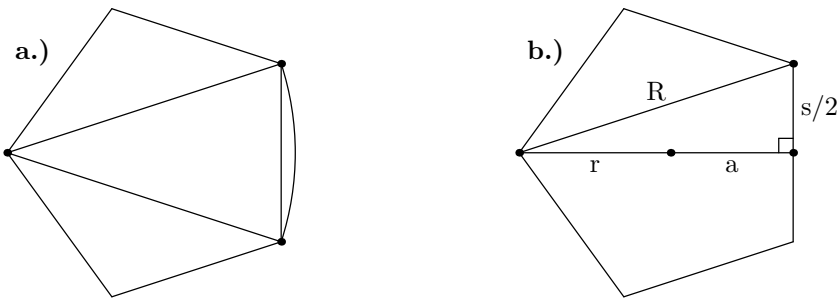
Abstract

Solids of constant width, or spheroforms, can be created by rotating Reuleaux polygons about an axis of symmetry. Reuleaux polygons are curves of constant width. For curves of constant width any two parallel lines that are tangent to the curve are always equidistant. Any two parallel planes tangent to the surface of a spheroform will maintain the same distance regardless of orientation. In this project we would like to find general formulas for the volume, surface area, centroid, and moments of inertia of these solids, depending on the number of sides and their width.

1 Constructing Reuleaux Polygons of Equal Width

Polygons with an odd number of sides contain vertices which are equidistant from two points on the opposite side. To create these Reuleaux polygons we sweep a minor arc centered at each of the vertices from one of the equidistant points to the other (*fig. a*). Since we want to make all of these polygons of equal width there must be some restriction on the side length because the distance between the vertices and the opposite equidistant points determines the width of the shape. From the figure below one can see that the distance from the point to the opposite side is $a + r$ where a is the apothem and r is the radius of the excircle. The hypotenuse of our triangle is R. This will ensure that the width of all the Reuleaux polygons will remain constant. The remaining leg will be of length 1/2 of the side length (*fig. b*).



From the right triangle in (*fig. b*) we have the equation:

$$R^2 = \left(\frac{s}{2}\right)^2 + (a+r)^2$$

To solve for s we substitute:

$$r = \frac{s\left(csc\left(\frac{\pi}{n}\right)\right)}{2}$$

where n is the number of sides. Likewise we can substitute:

$$a = \frac{s\left(cot\left(\frac{\pi}{n}\right)\right)}{2}$$

Therefore our equation becomes:

$$R^2 = \left(\frac{s}{2}\right)^2 + \left(\frac{s\left(cot\left(\frac{\pi}{n}\right)\right)}{2} + \frac{s\left(csc\left(\frac{\pi}{n}\right)\right)}{2}\right)^2$$

$$R^2 = \frac{s^2}{4}\left(1 + \left(cot\left(\frac{\pi}{n}\right)\right)^2 + \left(csc\left(\frac{\pi}{n}\right)\right)^2\right)$$

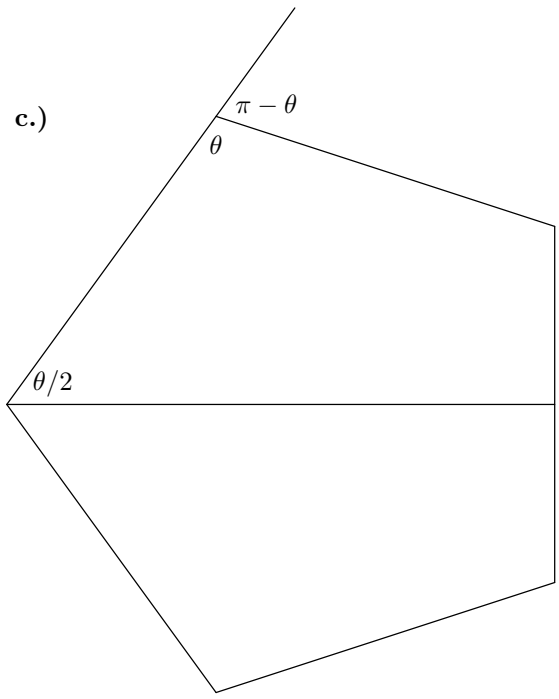
$$s = \sqrt{\frac{4R^2}{\left(1 + \left(cot\left(\frac{\pi}{n}\right)\right)^2 + \left(csc\left(\frac{\pi}{n}\right)\right)^2\right)}}$$

Now that we have s in terms of R and n we can determine the coordinates for each of the polygon's vertices. The internal angle of a regular polygon is given by the equation $\theta = \left(1 - \frac{2}{n}\right)\pi$. Assuming the leftmost vertex is at the origin in (*fig. c*) , $X_1 = 0$ and $Y_1 = 0$. Moving clockwise from the first vertex for $k > 1$ we have:

$$X_k = \sum_{k=2}^{k} s\left(cos\left(\frac{\theta}{2} - (k-2)(\pi - \theta)\right)\right)$$

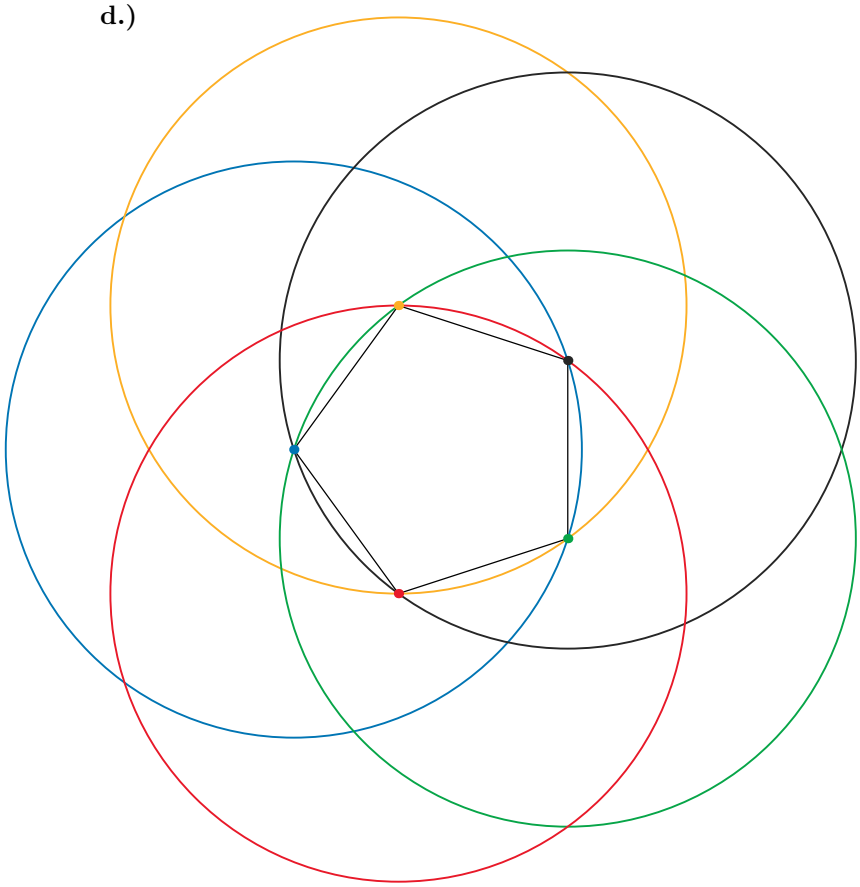
$$Y_k = \sum_{k=2}^{k} s\left(sin\left(\frac{\theta}{2} - (k-2)(\pi - \theta)\right)\right)$$

c.)



Using the coordinates obtained from the equations on the previous page we can now center a circle of radius R at each of the vertices as seen in (*fig. d*). The intersection of these circles is a regular Reuleaux polygon.

d.)

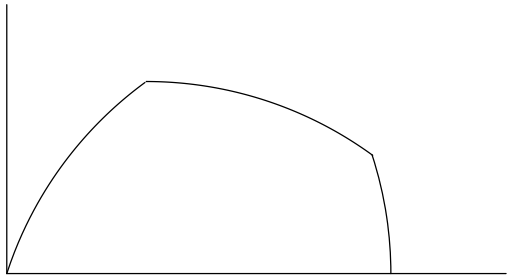


2 Rotating Reuleaux Polygons about an Axis of Symmetry

To create a Reuleaux spheroform we must rotate the shape in (*fig. d*) about the *x-axis*. Since our shape is symmetrical it is sufficient to use the upper half to create the piecewise function to be rotated. The upper half of the circle centered at the $\frac{n+3}{2}$ point will make up the first section of our piecewise function from X_1 to X_2 continuing clockwise all the way to the *n-th* point we can find each subsequent piece except the final one which is a special case. The last piece of our function will be bound by $X_{\frac{n+1}{2}}$ and R and is also the upper half of the circle centered at the origin. Our piecewise function is described by the equation below:

$$f_i(x) = \sqrt{R^2 - (x - X_{\frac{n+1}{2}+i})^2} + Y_{\frac{n+1}{2}+i} \text{ where } X_i \leq x \leq X_{i+1} \text{ for } i \text{ from 1 to } \frac{n-1}{2}$$

$$f_{\frac{n+1}{2}}(x) = \sqrt{R^2 - x^2} \text{ where } X_{\frac{n+1}{2}} \leq x \leq R$$



With these functions we can now rotate our Reuleaux polygon about the *x-axis* by letting the functions above define our radius as a function of x.

3 Physical Properties of Regular Reuleaux Spheroforms

3.1 Volume

To find the volume we integrate using the fact that the volume element $dV = rdrd\theta dx$. We get the following equation:

$$V(n,R) = \left(\sum_{i=1}^{\frac{n-1}{2}} \int_0^{2\pi} \int_{X_i}^{X_{i+1}} \int_0^{f_i(x)} rdrdxd\theta \right) + \left(\int_0^{2\pi} \int_{X_{\frac{n+1}{2}}}^{R} \int_0^{f_{\frac{n+1}{2}}(x)} rdrdxd\theta \right)$$

After integration we get the following equation:

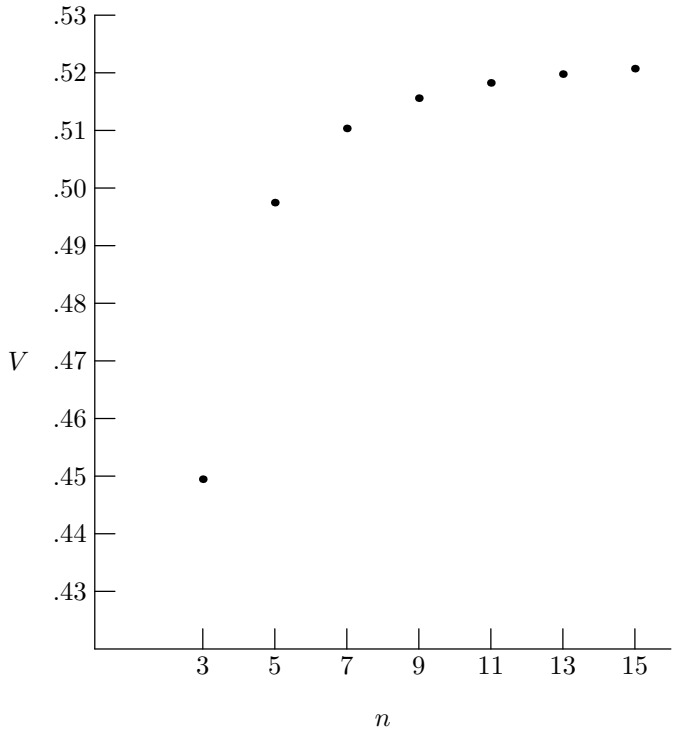
$$\begin{aligned} V(n,R) = \Bigg(\sum_{i=1}^{\frac{n-1}{2}} \pi \Bigg(& R^2x - \frac{1}{3}x^3 + x^2a - a^2x + bx\sqrt{R^2 - x^2 + 2xa - a^2} \\ & -ba\sqrt{R^2 - x^2 + 2xa - a^2} - bR^2 arctan\left(\frac{-x+a}{\sqrt{R^2 - x^2 + 2xa - a^2}} \right) + b^2 \Bigg) \Bigg|_c^d \Bigg) \\ & + \pi \left(R^2x - \frac{1}{3}x^3 \right) \Bigg|_e^R \end{aligned}$$

Where:

$$a = X_{\frac{n+1}{2}+i} \qquad b = Y_{\frac{n+1}{2}+i} \qquad c = X_i \qquad d = X_{i+1} \qquad e = X_{\frac{n+1}{2}}$$

All of the following equations are also exact integrals, and can be evaluated without numerical integration.

The graph below is a plot of the volume of a regular Reuleaux spheroform where $R = 1$ as a function of n.

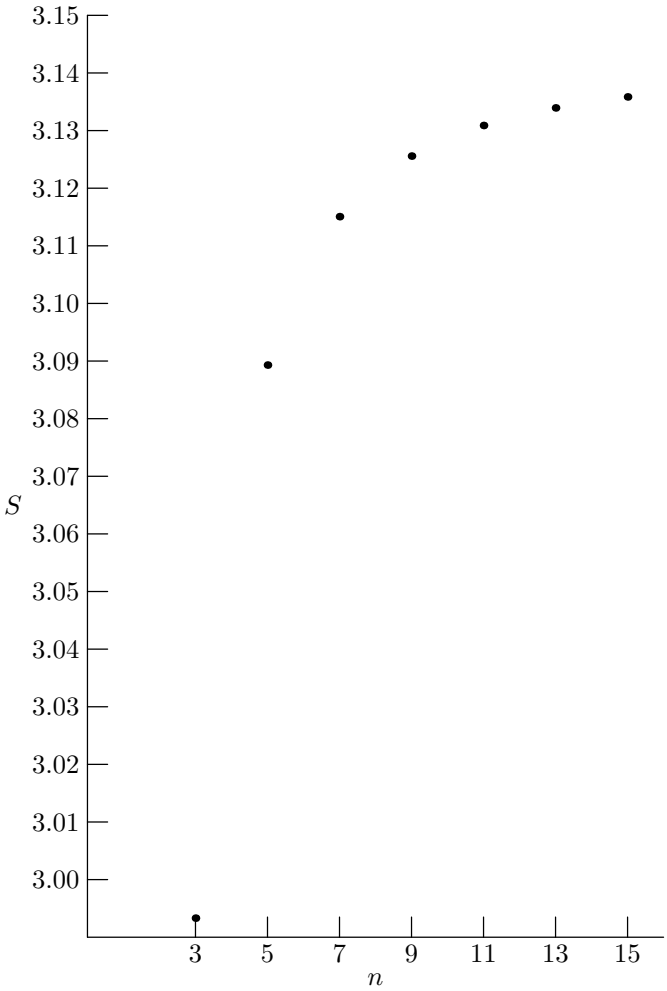


3.2 Surface Area

To find the surface area we integrate using the formula $dS = 2\pi f(x)\sqrt{1+[f'(x)]^2}dx$ for surfaces of revolution. Therefore:

$$S(n,R) = \left(\sum_{i=1}^{\frac{n-1}{2}} 2\pi \int_{X_i}^{X_{i+1}} f_i(x)\sqrt{1+[f_i'(x)]^2}dx\right) + \left(2\pi \int_{X_{\frac{n+1}{2}}}^{R} f_{\frac{n+1}{2}}(x)\sqrt{1+[f'_{\frac{n+1}{2}}(x)]^2dx}\right)$$

The graph below is a plot of the surface area of a regular Reuleaux spheroform where $R = 1$ as a function of n.



3.3 Center of Mass

To find the center of mass we use the equation $\int\int\int \bar{x} r dr dx d\theta = \int\int\int x r dr dx d\theta$. Since $\bar{x}$ is constant we can factor it out of the integral and solve.

$$\bar{x} = \frac{\int\int\int x r dr dx d\theta}{\int\int\int dV}$$

With the equation above we can find the center of mass of each piece of our spheroform. Since we can treat each of the pieces as a point mass located at the center of mass we can use the equation below to find the center of mass of the complete spheroform. Since our shape is radially symmetric our center of mass on the *y-axis z-axis* and is y=0 and z=0 respectively.

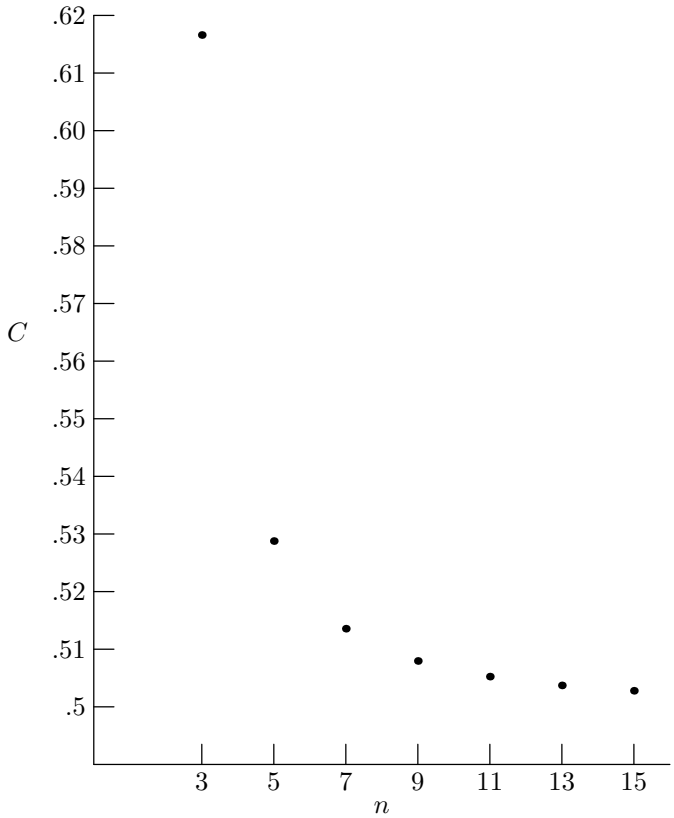
$$C = \frac{\sum m_j x_j}{\sum m_j}$$

Since we have a constant density:

$$C = \frac{\sum V_j x_j}{\sum V_j}$$

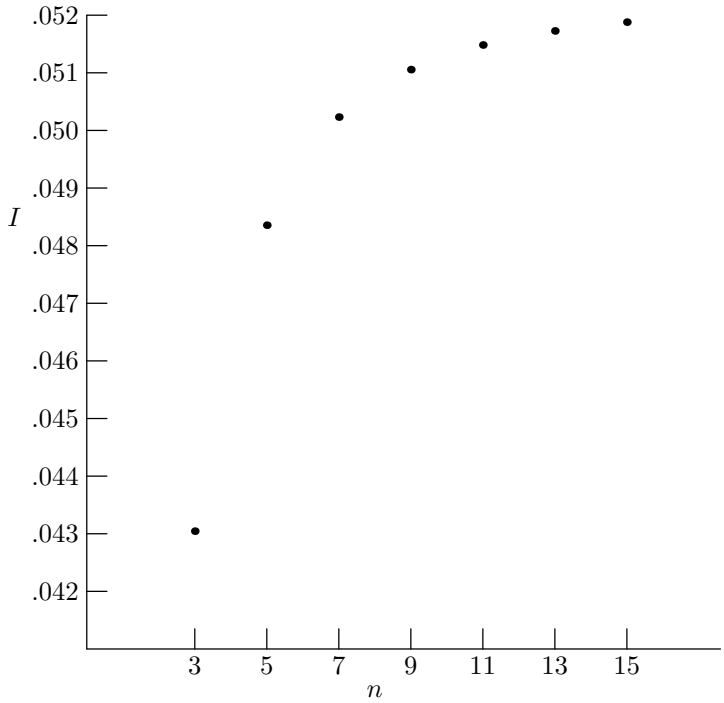
$$C(n,R) = \frac{\left(\sum_{i=1}^{\frac{n-1}{2}} \int_0^{2\pi} \int_{X_i}^{X_{i+1}} \int_0^{f_i(x)} r x dr dx d\theta\right) + \left(\int_0^{2\pi} \int_{X_{\frac{n+1}{2}}}^{R} \int_0^{f_{\frac{n+1}{2}}(x)} r x dr dx d\theta\right)}{\left(\sum_{i=1}^{\frac{n-1}{2}} \int_0^{2\pi} \int_{X_i}^{X_{i+1}} \int_0^{f_i(x)} r dr dx d\theta\right) + \left(\int_0^{2\pi} \int_{X_{\frac{n+1}{2}}}^{R} \int_0^{f_{\frac{n+1}{2}}(x)} r dr dx d\theta\right)}$$

The graph below is a plot of the center of mass of a regular Reuleaux spheroform where $R = 1$ as a function of n.



3.4 Moment of Inertia

To find the moment of inertia we use the equation $I = \int\int\int r^2 drdxd\theta$. We get the following equation:

$$I(n,R) = \left(\sum_{i=1}^{\frac{n-1}{2}} \int_0^{2\pi} \int_{X_i}^{X_{i+1}} \int_0^{f_i(x)} r^2 drdxd\theta\right) + \left(\int_0^{2\pi} \int_{X_{\frac{n+1}{2}}}^{R} \int_0^{f_{\frac{n+1}{2}}(x)} r^2 drdxd\theta\right)$$

The graph below is a plot of the moment of inertia along the x-*axis* of a regular Reuleaux spheroform where $R = 1$ as a function of n.